جواب سوال ۵ سری پنجم

اصلاحیه

$$\Psi(x,t)=\sum_{n=1}^{\infty} C_n u_n(x)\, e^{\frac{-iE_n t}{\hbar}}$$

$$C_n=\int_{-\frac{a}{2}}^{0} dx\, u_n^*(x)\,\Psi(x,0) \Rightarrow C_n=\int_{-\frac{a}{2}}^{0} dx \sqrt{\frac{2}{a}} \sin\frac{n\pi x}{a}\sqrt{\frac{2}{a}} = \frac{2}{a}\left(-\frac{a}{n\pi}\cos\frac{n\pi x}{a}\right)\Bigg|_{-\frac{a}{2}}^{0}$$

$$\boxed{C_n=-\frac{2}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)}$$

$$P_1=|C_1|^2 \Rightarrow P_1=\left|-\frac{2}{1\times\pi}\left(1-\cos\frac{1\times\pi}{2}\right)\right|^2 \Rightarrow P_1=\frac{4}{\pi^2}$$

$$P_2=|C_2|^2 \Rightarrow P_2=\left|-\frac{2}{2\times\pi}\left(1-\cos\frac{2\times\pi}{2}\right)\right|^2 \Rightarrow P_2=\frac{4}{\pi^2}$$

$$P_n=|C_n|^2 \to P_n=\left|-\frac{2}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)\right|^2 \Rightarrow P_n=\frac{4}{n^2\pi^2}\left(1-\cos\frac{n\pi}{2}\right)^2$$

$$n_1=1,3,5,\dots \to \cos\frac{n_1\pi}{2}=0 \to P_{n_1}=\frac{4}{n_1^2\pi^2}$$

$$n_2=2,6,10,\dots \to \cos\frac{n_2\pi}{2}=-1 \to P_{n_2}=\frac{16}{n_2^2\pi^2}$$

$$n_3=4,8,12,\dots \to \cos\frac{n_3\pi}{2}=1 \to P_{n_3}=0$$

$$\sum P_n=\sum_{n_1}P_{n_1}+\sum_{n_2}P_{n_2}+\sum_{n_3}P_{n_3} \Rightarrow \sum_{n=1}^{\infty}P_n=\sum_{n_1}\frac{4}{n_1^2\pi^2}+\sum_{n_2}\frac{16}{n_2^2\pi^2}+\sum_{n_3}0$$

$$=\frac{4}{\pi^2}\sum_{n_1}\frac{1}{n_1^2}+\frac{16}{\pi^2}\sum_{n_1}\frac{1}{(2n_1)^2}=\frac{4}{\pi^2}\sum_{n_1}\frac{1}{n_1^2}+\frac{4}{\pi^2}\sum_{n_1}\frac{1}{n_1^2} \Rightarrow \sum_{n=1}^{\infty}P_n=\frac{8}{\pi^2}\sum_{n_1}\frac{1}{n_1^2}$$

$$\to \sum P_n=\frac{8}{\pi^2}\times\frac{\pi^2}{8}=1$$

استفاده شد از $\longrightarrow \sum\frac{1}{n^2}=\sum_{\text{فرد}}\frac{1}{n^2}+\sum_{\text{زوج}}\frac{1}{n^2}$

$$\sum_{n}\frac{1}{n^2}=\sum_{\text{فرد}}\frac{1}{n^2}+\frac{1}{4}\sum\frac{1}{n^2}$$

$$\frac{\pi^2}{6}-\frac{1}{4}\,\frac{\pi^2}{6}=\sum_{\text{فرد}}\frac{1}{n^2}$$

$$\frac{\pi^2}{6}-\frac{\pi^2}{24}=\frac{3\pi^2}{24}=\frac{\pi^2}{8}$$

[redacted]

صلاحیه سوال ۳ سری پنجم

$$\Phi(p)=\int_0^a dx\sqrt{\frac{۲}{a}}\,\sin\frac{n\pi x}{a}\,\frac{e^{\frac{ipx}{\hbar}}}{\sqrt{2\pi\hbar}}$$

$$=\int_0^a dx\sqrt{\frac{۲}{a}}\,\frac{e^{\frac{i\pi x}{a}}-e^{-\frac{i\pi x}{a}}}{۲i}\,\frac{e^{\frac{ipx}{\hbar}}}{\sqrt{۲\pi\hbar}} \Rightarrow \Phi(p)=\frac{1}{۲i\sqrt{\pi\hbar a}}\int_0^a dx\left[e^{ix\left(\frac{n\pi}{a}+\frac{p}{\hbar}\right)}-e^{ix\left(-\frac{n\pi}{a}+\frac{p}{\hbar}\right)}\right]$$

$$\Phi(p)=\frac{1}{۲i\sqrt{\pi\hbar a}}\left[\frac{1}{i\left(\frac{n\pi}{a}+\frac{p}{\hbar}\right)}e^{ix\left(\frac{n\pi}{a}+\frac{p}{\hbar}\right)}-\frac{1}{i\left(-\frac{n\pi}{a}+\frac{p}{\hbar}\right)}e^{ix\left(-\frac{n\pi}{a}+\frac{p}{\hbar}\right)}\right]\Bigg|_0^a$$

$$=\frac{1}{\sqrt{۴\pi\hbar a}}\left[\frac{1}{\frac{n\pi}{a}+\frac{p}{\hbar}}e^{in\pi+\frac{ipa}{\hbar}}-\frac{1}{-\frac{n\pi}{a}+\frac{p}{\hbar}}e^{-in\pi+\frac{ipa}{\hbar}}-\frac{1}{\frac{n\pi}{a}+\frac{p}{\hbar}}+\frac{1}{-\frac{n\pi}{a}+\frac{p}{\hbar}}\right]$$

$$\rightarrow\Phi(p)=\frac{1}{\sqrt{۴\pi\hbar a}}\left[\frac{1}{\frac{n\pi}{a}+\frac{p}{\hbar}}\left(e^{in\pi}e^{\frac{ipa}{\hbar}}-1\right)-\frac{1}{-\frac{n\pi}{a}+\frac{p}{\hbar}}\left(e^{-in\pi}e^{\frac{ipa}{\hbar}}-1\right)\right]\Rightarrow$$

$$\Phi(p)=\frac{1}{\sqrt{۴\pi\hbar a}}\left[\frac{1}{\frac{n\pi}{a}+\frac{p}{\hbar}}\left((-1)^n e^{\frac{ipa}{\hbar}}-1\right)+\frac{1}{\frac{n\pi}{a}-\frac{p}{\hbar}}\left((-1)^n e^{\frac{ipa}{\hbar}}-1\right)\right]$$

$$=\frac{1}{\sqrt{۴\pi\hbar a}}\,\frac{1}{\left(\frac{n\pi}{a}\right)^۲-\left(\frac{p}{\hbar}\right)^۲}\,\frac{۲n\pi}{a}\left[(-1)^n e^{\frac{ipa}{\hbar}}-1\right]$$

$$P(p)=|\Phi(p)|^۲=\frac{1}{۴\pi\hbar a}\,\frac{1}{\left[\left(\frac{n\pi}{a}\right)^۲-\left(\frac{p}{\hbar}\right)^۲\right]}\,\frac{۴n^۲\pi^۲}{a^۲}\left[(-1)^n e^{-\frac{ipa}{\hbar}}-1\right]\left[(-1)^n e^{\frac{ipa}{\hbar}}-1\right]$$

$$=\frac{n^۲\pi^۲}{\hbar a^۲}\,\frac{1}{\left[\left(\frac{n\pi}{a}\right)^۲-\left(\frac{p}{\hbar}\right)^۲\right]^۲}\left[1-(-1)^n e^{-\frac{ipa}{\hbar}}-(-1)^n e^{\frac{ipa}{\hbar}}+1\right]$$

$$=\frac{n^۲\pi}{\hbar a^۲}\,\frac{1}{\left[\left(\frac{n\pi}{a}\right)^۲-\left(\frac{p}{\hbar}\right)^۲\right]^۲}\left[۲-(-1)^n\,۲\cos\frac{pa}{\hbar}\right]=\frac{۲n^۲\pi}{\hbar a^۳}\,\frac{1}{\left[\left(\frac{n\pi}{a}\right)^۲-\left(\frac{p}{\hbar}\right)^۲\right]^۲}\left[1-(-1)^n\cos\frac{pa}{\hbar}\right]$$

$$\sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{۲i}$$

$$\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{۲}$$

مسئله ۶.۴ کتاب گاسیوروویچ، ویرایش سوم

$$\frac{2m}{\hbar^2}V(x)=\frac{\lambda}{a}\delta(x-b) \longrightarrow$$ باتوجه به پتانسیل، ذره بجز در $x=b$ ذره ی آزاد است.

$$u(x)=\begin{cases} Ae^{ikx}+Be^{-ikx} & x<b \\ Ce^{ikx}+De^{-ikx} & x>b \end{cases}$$

با توجه به پیوستگی توابع موج در $x=b$ $\longmapsto Ae^{ikx}+Be^{-ikx}=Ce^{ikx}+De^{-ikx}$ (1)

شرط مشتق $\longmapsto \left(\frac{du}{dx}\right)_{b+\epsilon}-\left(\frac{du}{dx}\right)_{b-\epsilon}=\frac{2m}{\hbar^2}V_0\,u(b) \Rightarrow (ikCe^{ikb}-ikDe^{-ikb})-(ikAe^{ikb}-ikBe^{-ikb})=\frac{\lambda}{a}u(b)$ (2)

تعریف $\rightarrow \alpha=Ae^{ikb}$ و $\beta=Be^{-ikb}$ و $\gamma=Ce^{ikb}$ ، $\theta=De^{-ikb}$

$$(1)\longmapsto \alpha+\beta=\gamma+\theta \Longrightarrow \beta=\gamma+\theta-\alpha \quad (3)$$

$$(2)\longmapsto ik(\gamma-\theta)-ik(\alpha-\beta)=\frac{\lambda}{a}u(b) \Longrightarrow \gamma-\theta-\alpha+\beta=\frac{\lambda}{ika}u(b)$$

$$\gamma-\theta-\alpha+\beta=\frac{\lambda}{ika}(Ae^{ikb}+Be^{-ikb}) \Rightarrow \gamma-\theta-\alpha+\beta=\frac{\lambda}{ika}(\alpha+\beta)$$

$$(3)\longmapsto \gamma-\theta-\alpha+\gamma-\alpha=\frac{\lambda}{ika}(\alpha+\gamma+\theta-\alpha) \Rightarrow 2\gamma-2\alpha=\frac{\lambda}{ika}\gamma+\frac{\lambda}{ika}\theta \Rightarrow \left(2-\frac{\lambda}{ika}\right)\gamma=2\alpha+\frac{\lambda}{ika}\theta$$

$$\rightarrow \frac{2ika-\lambda}{ika}\gamma=2\alpha+\frac{\lambda}{ika}\theta \Rightarrow \gamma=\frac{2ika}{2ika-\lambda}\alpha+\frac{\lambda}{2ika-\lambda}\theta$$

$$Ce^{ikb}=\frac{2ika}{2ika-\lambda}Ae^{ikb}+\frac{\lambda}{2ika-\lambda}De^{-ikb} \Longrightarrow \boxed{C=\frac{2ika}{2ika-\lambda}A+\frac{\lambda e^{-2ikb}}{2ika-\lambda}D} \quad (4)$$

جاگذاری (۴) در (۳) $\longmapsto \beta=\gamma+\theta-\alpha \Rightarrow Be^{-ikb}=Ce^{ikb}+De^{-ikb}-Ae^{ikb} \Rightarrow B=Ce^{2ikb}+D-Ae^{2ikb}$

$$B=\frac{2ika}{2ika-\lambda}e^{2ikb}A+\frac{\lambda}{2ika-\lambda}D+D-Ae^{2ikb} \Rightarrow \boxed{B=\frac{\lambda e^{2ikb}}{2ika-\lambda}A+\frac{2ika}{2ika-\lambda}D}$$

$$S_{11}=\frac{2ika}{2ika-\lambda} \qquad S_{12}=\frac{\lambda e^{-2ikb}}{2ika-\lambda} \qquad S_{21}=\frac{\lambda e^{2ikb}}{2ika-\lambda} \qquad S_{22}=\frac{2ika}{2ika-\lambda}$$

$$S=\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22}\end{pmatrix} \Longrightarrow S=\begin{pmatrix} \frac{2ika}{2ika-\lambda} & \frac{\lambda}{2ika-\lambda}e^{-2ikb} \\ \frac{\lambda}{2ika-\lambda}e^{2ikb} & \frac{2ika}{2ika-\lambda}\end{pmatrix}$$

برای نشان دادن یکانی بودن این ماتریس
از مسئله ۱.۴ استفاده شده است (مسئله همین فصل)

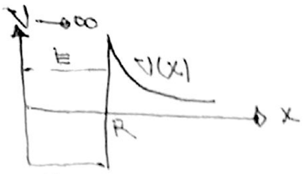
$$|S_{11}|^2+|S_{21}|^2 \rightarrow \frac{4k^2a^2}{4k^2a^2+\lambda^2}+\frac{\lambda^2}{4k^2a^2+\lambda^2}=1 \quad , \quad |S_{12}|^2+|S_{22}|^2=\frac{\lambda^2}{4k^2a^2+\lambda^2}+\frac{4k^2a^2}{4k^2a^2+\lambda^2}=1$$

$$S_{11}S_{12}^*+S_{21}S_{22}^*=\frac{2ika}{2ika-\lambda}\times\frac{\lambda}{-2ika-\lambda}e^{2ikb}+\frac{\lambda e^{2ib}}{2ika-\lambda}\times\frac{-2ika}{-2ika-\lambda}=0$$ ماتریس حاصل یکانی است

$$2ika-\lambda=0 \Rightarrow 2ika=\lambda \Rightarrow ik=\frac{\lambda}{2a} \rightarrow k=\frac{\lambda}{2a}$$

عناصر ماتریس به ازای $2ika-\lambda=0$ بی‌نهایت می‌شود

مسأله ۸.۴ کتاب گسیورویچ، ویرایش سوم



$$V(x) = \frac{\hbar^2 l(l+1)}{2mx^2} \qquad x > R$$

$$|T|^2 \cong e^{-2\int_A^B dx \sqrt{\frac{2m}{\hbar^2}[V(x)-E]}} \Longrightarrow e^{-S} = e^{-2\int_A^B dx \sqrt{\frac{2m}{\hbar^2}[V(x)-E]}}$$

$$S = 2\int_R^b dx \sqrt{\frac{2m}{\hbar^2}\left[\frac{\hbar^2 l(l+1)}{2mx^2} - E\right]} \Longrightarrow S = \frac{2}{\hbar}\int_R^b dx \sqrt{\frac{\hbar^2 l(l+1)}{x^2} - 2mE} \qquad (1)$$

مقدار b مقداری است که زیر رادیکال برابر صفر شود $\longmapsto \frac{\hbar^2 l(l+1)}{b^2} - 2mE = 0 \Rightarrow b^2 = \frac{\hbar^2 l(l+1)}{2mE} \Rightarrow b = \sqrt{\frac{\hbar^2 l(l+1)}{2mE}} \Rightarrow b = \frac{1}{k}\sqrt{l(l+1)}$

$$k^2 = \frac{2mE}{\hbar^2}$$

$$(1) \longrightarrow S = \frac{2}{\hbar}\int_R^b \sqrt{\frac{\hbar^2 l(l+1)}{x^2}}\, dx \sqrt{1 - \frac{2mE}{\hbar^2 l(l+1)}x^2} \Longrightarrow S = 2\int_R^b \frac{dx}{x}\sqrt{l(l+1)}\sqrt{1 - \frac{2mE}{\hbar^2 l(l+1)}x^2}$$

تغییر متغیر $\longmapsto \sqrt{\frac{2mE}{\hbar^2 l(l+1)}}\,x = u \longrightarrow \sqrt{\frac{2mE}{\hbar^2 l(l+1)}}\,dx = du \Rightarrow \frac{dx}{x} = \frac{du}{u}$

$$S = 2\sqrt{l(l+1)}\int_{\frac{R}{d}}^{1} \frac{du}{u}\sqrt{1-u^2} \Longrightarrow S = 2\sqrt{l(l+1)}\left[\sqrt{1-u^2} - \ln\frac{1+\sqrt{1-u^2}}{u}\right]\Bigg|_{\frac{R}{b}}^{1}$$

$$S = 2\sqrt{l(l+1)}\left[0 - 0 - \sqrt{1-\left(\frac{R}{b}\right)^2} + \ln\frac{1+\sqrt{1-\left(\frac{R}{b}\right)^2}}{\frac{R}{b}}\right] = 2\sqrt{l(l+1)}\left[\ln\frac{1+\sqrt{1-\left(\frac{R}{d}\right)^2}}{\frac{R}{b}} - \sqrt{1-\left(\frac{R}{b}\right)^2}\right]$$

$$l \gg 1 \Rightarrow b \cong \frac{l}{k} \Rightarrow \frac{R}{b} \cong \frac{kR}{l} \Rightarrow S = 2l\left[\ln\frac{1+\sqrt{1-\left(\frac{kR}{l}\right)^2}}{\frac{kR}{l}} - \sqrt{1-\left(\frac{kR}{l}\right)^2}\right]$$

$$|T|^2 \cong e^{-S} \Rightarrow |T|^2 = \left[\frac{1+\sqrt{1-\left(\frac{kR}{l}\right)^2}}{\frac{kR}{l}}\right]^{-2l} e^{-2l\sqrt{1-\frac{kR}{l}}} \longrightarrow |T|^2 = \left(\frac{kR}{l}\right)^{2l} e^{-2l}$$

مسأله ۱۰.۴ گاسیروویچ، ویرایش سوم

الف) برای حالتهای مقید جواب در ناحیه $x>a$ تابع موج به صورت $u(x)=Ae^{-\alpha x}$ و $\alpha=\frac{\sqrt{2mE_B}}{\hbar}$

$$\frac{1}{u}\frac{du}{dx}\Big|_{x=a}=f(E)\Rightarrow\frac{1}{Ae^{-\alpha x}}\left(-\alpha Ae^{-\alpha x}\right)\Big|_{x=a}=f(E)\Rightarrow-\alpha=f(E)$$

$$\Rightarrow-\frac{\hbar}{\sqrt{2mE_B}}=f(E)\Rightarrow\frac{\hbar^2}{2mE_B}=\left[f(E_B)\right]^2$$

ب) $u(x)=e^{-ikx}+Re^{ikx}$ ، $f(E)\equiv f_0$

$$\frac{1}{u}\frac{du}{dx}\Big|_{x=a}=f(E)\Rightarrow\frac{1}{e^{-ikx}+Re^{ikx}}\left(-ike^{-ikx}+ikRe^{ikx}\right)\Big|_{x=a}=f_0$$

$$\Rightarrow\frac{ikRe^{ika}-ike^{-ika}}{e^{-ika}+Re^{ika}}=f_0\Rightarrow ikRe^{ika}-ike^{-ika}=f_0e^{-ika}+f_0Re^{ika}$$

$$\Rightarrow Re^{ika}(ik-f_0)=e^{-ika}(ik+f_0)\Rightarrow R=\frac{ik+f_0}{ik-f_0}e^{-2ika}$$

$$|R|^2=RR^*\Rightarrow|R|^2=\left(\frac{ik+f_0}{ik-f_0}e^{-2ika}\right)\times\left(\frac{-ik+f_0}{-ik-f_0}e^{2ika}\right)\Rightarrow|R|^2=\frac{k^2+f_0^2}{k^2+f_0^2}\Rightarrow|R|^2=1$$

مسأله ۱۱.۴ گاسیروویچ، ویرایش سوم

با توجه به تقارن پتانسیل حول $x=0$ توابع موج را برای نقاط a و b بررسی می‌کنیم

جوابهای زوج

$$u(x)=\begin{cases}\cosh\alpha x & 0<x<b\\ A\sin qx+B\cos qx & b<x<a\\ Ce^{-\alpha x} & x>a\end{cases}\quad,\quad\begin{cases}\alpha^2=\frac{2m}{\hbar^2}|E|\\ q^2=\frac{2m}{\hbar^2}(V-E)\end{cases}$$

با توجه مسئله قبل که داشتیم $\left(\frac{1}{u}\right)\left(\frac{du}{dx}\right)$ برای نقاط $x=b$ و $x=a$ می‌توان نوشت:

$x=b$

$$\frac{1}{\cosh\alpha x}\alpha\sinh\alpha x\Big|_{x=b}=\frac{1}{A\sin qx+B\cos qx}\left(Aq\cos qx-Bq\sin qx\right)\Big|_{x=b}$$

$$\Rightarrow\alpha\tanh\alpha b=q\,\frac{A\cos qb-B\sin qb}{A\sin qb+B\cos qb}\Rightarrow B(\alpha\cos qb\tanh\alpha b+q\sin qb)=A(q\cos qb-\alpha\sin qb\tanh\alpha b)\quad(1)$$

$x=a$

$$\frac{1}{A\sin qx+B\cos qx}\left(Aq\cos qx-Bq\sin qx\right)\Big|_{x=b}=\frac{1}{Ce^{-\alpha x}}\left(-\alpha Ce^{-\alpha x}\right)\Big|_{x=a}\Rightarrow$$

$$\frac{Aq\cos qa-Bq\sin qa}{A\sin qa+B\cos qa}=-\alpha\Rightarrow B(\alpha\cos qa-q\sin qa)=-A(q\cos qa+\alpha\sin qa)\quad(2)$$

① از $\longmapsto$ $$\frac{B}{A}=\frac{q\cos qb-\alpha\tanh\alpha b\,\sin qb}{q\sin qb+\alpha\tanh\alpha b\,\cos qb}$$

② از $\longmapsto$ $$\frac{B}{A}=\frac{q\cos qa+\alpha\sin qa}{q\sin qa-\alpha\cos qa}$$

با مساوی قرار دادن این دو و طرفین و وسطین کردن سپس از اندک عملیات جبری

$$q^2\sin q(a-b)-\alpha\cos q(a-b)=\alpha\tanh\alpha b\left[\alpha\sin q(a-b)+q\cos q(a-b)\right]$$

$$\rightarrow \underbrace{\frac{\sin q(a-b)}{\cos q(a-b)}}_{\tan q(a-b)}=\frac{\alpha q(\tanh\alpha b+1)}{q^2-\alpha^2\tanh\alpha b}$$

جواب های فرد:

$$u(x)=\begin{cases}\sinh\alpha x & 0<x<b\\ A\sin qx+B\cos qx & b<x<a\\ Ce^{-\alpha x} & x>a\end{cases}\qquad ,\qquad \begin{cases}\alpha^2=\frac{2m}{\hbar^2}|E|\\ q^2=\frac{2m}{\hbar^2}(V-E)\end{cases}$$

$x=b$ $$\frac{1}{\sinh\alpha x}\alpha\cosh\alpha x\Big|_{x=b}=\frac{1}{A\sin qx+B\cos qx}(Aq\cos qx-Bq\sin qx)\Big|_{x=b}\Rightarrow$$

$$\alpha\coth\alpha b=q\,\frac{A\cos qb-B\sin qb}{A\sin qb+B\cos qb}\Longrightarrow B(\alpha\cos qb\coth\alpha b+q\sin qb)=A(q\cos qb-\alpha\sin qb\tanh\alpha b)\quad ③$$

مشابه گذشته

برای نقطه a با توجه به اینکه توابع موج عوض شده اند بین رابطه ② عوض می شود.

② و ③ از $\longmapsto$ $$\frac{\sin q(a-b)}{\cos q(a-b)}=\frac{\alpha q(1+\coth\alpha b)}{q^2-\alpha^2\coth\alpha b}\Longrightarrow \tan q(a-b)=\frac{\alpha q(1+\coth\alpha b)}{q^2-\alpha^2\coth\alpha b}$$

قضیه ویریال $\mapsto$ $\left\langle \frac{p^2}{2m} \right\rangle = \frac{1}{2}\left\langle x\frac{dV}{dx}\right\rangle$ — مسئله ۱۴۰۴ سیدروح

$$\left\langle x\frac{dV(x)}{dx}\right\rangle = \int_{-\infty}^{\infty} dx\, \Psi(x)\, x\frac{dV(x)}{dx}\,\Psi(x)$$

$$= \int_{-\infty}^{\infty} dx\left[\frac{d}{dx}(\Psi^2 xV) - 2\Psi\frac{d\Psi}{dx}xV - \Psi^2 V\right]$$

جمله اول حذف می‌شه چون Ψ به صفر میل می‌کند

$$-2\int_{-\infty}^{\infty} dx\frac{d\Psi}{dx}xV\Psi = -2\int_{-\infty}^{\infty} dx\frac{d\Psi}{dx}x\left(E + \frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\Psi$$

$$\Rightarrow \boxed{\left\langle x\frac{dV}{dx}\right\rangle = -\langle V\rangle - 2E\int_{-\infty}^{+\infty} dx\frac{d\Psi}{dx}x\Psi - \frac{\hbar^2}{\cancel{2}m}\int_{-\infty}^{\infty} dx\frac{d\Psi}{dx}x\frac{d^2\Psi}{dx^2}} \quad (1)$$

انتگرال اول از جزء به جزء $\mapsto$

$$\int_{-\infty}^{+\infty} dx\frac{d\Psi}{dx}x\Psi = \Psi x\Psi\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx\,\Psi\frac{d}{dx}(x\Psi) = 0 - \underbrace{\int_{-\infty}^{\infty} dx\,\Psi\left(\Psi + x\frac{d\Psi}{dx}\right)}$$

$$\boxed{-\int_{-\infty}^{\infty} dx\,\Psi^2 = -1}$$

$$\int_{-\infty}^{+\infty} dx\frac{d\Psi}{dx}x\Psi = -\int_{-\infty}^{\infty} dx\frac{d\Psi}{dx}x\Psi \Rightarrow 2\int_{-\infty}^{+\infty} dx\frac{d\Psi}{dx}x\Psi = -1$$

$$\Rightarrow \boxed{\int_{-\infty}^{+\infty} dx\frac{d\Psi}{dx}x\Psi = -\frac{1}{2}} \quad (2)$$

انتگرال دوم از روش جزء به جزء $\mapsto$

$$-\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} dx\frac{d\Psi}{dx}x\frac{d^2\Psi}{dx^2} = -\frac{\hbar^2}{2m}\left[\frac{d\Psi}{dx}x\frac{d\Psi}{dx}\right]\Bigg|_{-\infty}^{+\infty} + \frac{\hbar^2}{2m}\int_{-\infty}^{+\infty}\frac{d}{dx}\left(x\frac{d\Psi}{dx}\right)\frac{d\Psi}{dx}$$

جزء به جزء — اول دو تساوی

$$// \quad = 0 + \frac{\hbar^2}{2m}\int_{-\infty}^{\infty} dx\frac{d\Psi}{dx}\frac{d\Psi}{dx} + \boxed{\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} dx\frac{d\Psi}{dx}x\frac{d^2\Psi}{dx^2}}$$

اون طرف تساوی

$$-\frac{2\hbar^2}{2m}\int_{-\infty}^{+\infty} x\frac{d\Psi}{dx}x\frac{d^2\Psi}{dx^2} = \frac{\hbar^2}{2m}\left[\Psi\frac{d\Psi}{dx}\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} dx\,\Psi\frac{d^2\Psi}{dx^2}\right]$$

$$// \quad = -\frac{\hbar^2}{2m}\times 0 - \frac{\hbar^2}{2m}\int_{-\infty}^{+\infty} dx\,\Psi\frac{d^2\Psi}{dx^2}$$

$$// \quad = \frac{1}{2m}\int_{-\infty}^{\infty} dx\,\Psi\left(-i\hbar\frac{d}{dx}\right)^2\Psi$$

$$\rightarrow \boxed{-\frac{\hbar^2}{2m}\int_{-\infty}^{+\infty} x\frac{d\Psi}{dx}x\frac{d^2\Psi}{dx^2} = \left\langle\frac{p^2}{2m}\right\rangle} \quad (3)$$

$$(2)\ ,\ (3) \xrightarrow{\text{جایگذاری}} (1) \;\Longmapsto\; \left\langle x\frac{dV}{dx}\right\rangle = -\langle V\rangle - 2E\times\left(-\frac{1}{2}\right) + \left\langle\frac{p^2}{2m}\right\rangle$$

$$\left\langle x\frac{dV}{dx}\right\rangle = E - \langle V\rangle + \left\langle\frac{p^2}{2m}\right\rangle$$

$$\left\langle x\frac{dV}{dx}\right\rangle = \left\langle\frac{p^2}{2m}\right\rangle + \langle V\rangle - \langle V\rangle + \left\langle\frac{p^2}{2m}\right\rangle = 2\left\langle\frac{p^2}{2m}\right\rangle$$

$$\rightarrow \boxed{\left\langle\frac{p^2}{2m}\right\rangle = \frac{1}{2}\left\langle x\frac{dV}{dx}\right\rangle}$$

time independent schrodinger equation $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+V\psi=E\psi$

جواب سؤال V

for harmonic oscillator $V(x)=\frac{1}{2}m\omega^2x^2$

$$\rightarrow -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+\frac{1}{2}m\omega^2x^2\psi=E\psi$$

الف) $\Psi(x,t)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega}{2\hbar}\left(x^2+\frac{a^2}{2}(1+e^{-2i\omega t})+\frac{i\hbar t}{m}-2axe^{-i\omega t}\right)\right]$

$$\frac{\partial\psi}{\partial t}=\left(-\frac{m\omega}{2\hbar}\right)\left[\frac{a^2}{2}(-2i\omega e^{-2i\omega t})+\frac{i\hbar}{m}-2ax(-i\omega)e^{-i\omega t}\right]\psi$$

$$i\hbar\frac{\partial\psi}{\partial t}=\left[-\frac{1}{2}ma^2\omega^2e^{-2i\omega t}+\frac{1}{2}\hbar\omega+max\omega^2e^{-i\omega t}\right]\psi$$

$$\frac{\partial\psi}{\partial x}=\left[\left(-\frac{m\omega}{2\hbar}\right)(2x-2ae^{-i\omega t})\right]\psi=-\frac{m\omega}{\hbar}(x-ae^{-i\omega t})\psi$$

$$\frac{\partial^2\psi}{\partial x^2}=-\frac{m\omega}{\hbar}\psi-\frac{m\omega}{\hbar}(x-ae^{-i\omega t})\frac{\partial\psi}{\partial x}=\left[-\frac{m\omega}{\hbar}+\left(\frac{m\omega}{\hbar}\right)^2(x-ae^{-i\omega t})^2\right]\psi$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+\frac{1}{2}m\omega^2x^2\psi=-\frac{\hbar^2}{2m}\left[-\frac{m\omega}{\hbar}+\left(\frac{m\omega}{\hbar}\right)^2(x-ae^{-i\omega t})^2\right]\psi+\frac{1}{2}m\omega^2x^2\psi$$

$$\text{''}\quad=\left[\frac{1}{2}\hbar\omega-\frac{1}{2}m\omega^2(x^2-2axe^{-i\omega t}+a^2e^{-2i\omega t})+\frac{1}{2}m\omega^2x^2\right]\psi$$

$$\text{''}\quad=\left[\frac{1}{2}\hbar\omega+max\omega^2e^{-i\omega t}-\frac{1}{2}m\omega^2a^2e^{-2i\omega t}\right]\psi$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+\frac{1}{2}m\omega^2x^2\psi=i\hbar\frac{\partial\psi}{\partial t}$$ time dependent schrodinger equation

ب) $|\psi|^2=\psi^*\psi=\sqrt{\frac{m\omega}{\pi\hbar}}\,e^{-\frac{m\omega}{2\hbar}\left[\left(x^2+\frac{a^2}{2}(1+e^{2i\omega t})-\frac{i\hbar t}{m}-2axe^{i\omega t}\right)+\left(x^2+\frac{a^2}{2}(1+e^{-2i\omega t})+\frac{i\hbar t}{m}-2axe^{-i\omega t}\right)\right]}$

$$=\sqrt{\frac{m\omega}{\pi\hbar}}\,e^{-\frac{m\omega}{2\hbar}\left[2x^2+a^2+a^2\cos(2\omega t)-4ax\cos(\omega t)\right]}$$

$a^2[1+\cos(2\omega t)]=2a^2\cos^2\omega t$

$$=\sqrt{\frac{m\omega}{\pi\hbar}}\,e^{-\frac{m\omega}{\hbar}\left[x^2-2ax\cos(\omega t)+a^2\cos^2(\omega t)\right]}=\sqrt{\frac{m\omega}{\pi\hbar}}\,e^{-\frac{m\omega}{\hbar}(x-a\cos\omega t)^2}$$

$y\equiv x-a\cos\omega t\rightarrow x=y+a\cos\omega t$

$$\langle x\rangle=\int x|\psi|^2dx=\int(y+a\cos\omega t)|\psi|^2dy=0+a\cos\omega t\int|\psi|^2dy=a\cos\omega t$$

$$\langle p\rangle=m\frac{d\langle x\rangle}{dt}=-ma\omega\sin\omega t.\quad \frac{d\langle p\rangle}{dt}=-ma\omega^2\cos\omega t,\quad V=\frac{1}{2}m\omega^2x^2\rightarrow\frac{dV}{dx}=m\omega^2x$$

$$\left\langle-\frac{dV}{dx}\right\rangle=-m\omega^2\langle x\rangle=-m\omega^2a\cos\omega t=\frac{d\langle p\rangle}{dt}$$

جواب سؤال ۸

$$H\psi = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \frac{\hbar^2 a^2}{m}\operatorname{sech}^2(ax)\,\psi$$

$$\frac{d\psi}{dx} = -Na\operatorname{sech}(ax)\tanh(ax) \quad ; \quad \frac{d^2\psi}{dx^2} = -Na^2\left[-\operatorname{sech}(ax)\tanh^2(ax) + \operatorname{sech}(ax)\operatorname{Sech}^2(ax)\right]$$

$$H\psi = \frac{\hbar^2}{2m}Na^2\left[-\operatorname{sech}(ax)\tanh^2(ax) + \operatorname{Sech}^3(ax)\right] - \frac{\hbar^2 a^2}{m}N\operatorname{sech}^3(ax)$$

$$= \frac{\hbar^2 a^2 N}{2m}\left[-\operatorname{Sech}(ax)\tanh^2(ax) + \operatorname{sech}^3(ax) - 2\operatorname{Sech}^3(ax)\right] = -\frac{\hbar^2 a^2}{2m}N\operatorname{sech}(ax)\left[\tanh^2(ax) + \operatorname{sech}^2(ax)\right]$$

But $\rightarrow \tanh^2\theta + \operatorname{sech}^2\theta = \dfrac{\sinh^2\theta}{\cosh^2\theta} + \dfrac{1}{\cosh^2\theta} = \dfrac{\sinh^2\theta + 1}{\cosh^2\theta} = 1$

$$\Rightarrow H\psi = -\frac{\hbar^2 a^2}{2m}\psi \longrightarrow \boxed{E = -\frac{\hbar^2 a^2}{2m}}$$

$$1 = |N|^2\int_{-\infty}^{\infty}\operatorname{sech}^2(ax)\,dx = |N|^2\frac{1}{a}\tanh(ax)\Big|_{-\infty}^{\infty} = \frac{2}{a}|N|^2 \Rightarrow \boxed{N = \sqrt{\frac{a}{2}}}$$

ج) $\dfrac{du_k}{dx} = \dfrac{N}{ik+a}\left[(ik - a\tanh ax)ik - a^2\operatorname{sech}^2 ax\right]e^{ikx}$

$$\frac{d^2u_k}{dx^2} = \frac{N}{ik+a}\left\{ik\left[(ik - a\tanh ax)ik - a^2\operatorname{sech}^2 ax\right] - a^2 ik\operatorname{sech}^2 ax + 2a^3\operatorname{sech}^2 ax\tanh ax\right\}e^{ikx}$$

$$-\frac{\hbar^2}{2m}\frac{d^2u_k}{dx^2} + Vu_k = \frac{N}{ik+a}\left\{-\frac{\hbar^2 ik}{2m}\left[-k^2 - iak\tanh ax - a^2\operatorname{sech}^2 ax\right] + \frac{\hbar^2 a^2}{2m}ik\operatorname{sech}^2 ax\right.$$
$$\left. - \frac{\hbar^2 a^3}{m}\operatorname{sech}^2 ax\tanh ax - \frac{\hbar^2 a^2}{m}\operatorname{sech}^2 ax\,(ik - a\tanh ax)\right\}e^{ikx}$$

$$= \frac{Ne^{ikx}}{ik+a}\frac{\hbar^2}{2m}\left(ik^3 - ak^2\tanh ax + ia^2k\operatorname{sech}^2 ax + ia^2k\operatorname{sech}^2 ax - 2a^3\operatorname{sech}^2 ax\tanh ax\right.$$
$$\left. - 2ia^2k\operatorname{sech}^2 ax + 2a^3\operatorname{sech}^2 ax\tanh ax\right)$$

$$= \frac{Ne^{ikx}}{ik+a}\frac{\hbar^2}{2m}k^2(ik - a\tanh ax) = \frac{\hbar^2k^2}{2m}u_k = Eu_k$$

$x \to \infty$, $\tanh ax \to +1 \to u_k(x) \to A\left(\dfrac{ik-a}{ik+a}\right)e^{ikx}$

$R = 0 \qquad T = \left|\dfrac{ik-a}{ik+a}\right|^2 = \left(\dfrac{-ik-a}{-ik+a}\right)\left(\dfrac{ik-a}{ik+a}\right) = 1$

for scattering from the left $(B=0)$, $\psi(x) = \begin{cases} Ae^{ikx} + B(\circ) \\ A\left(\dfrac{ik-a}{ik+a}\right)e^{ikx} & x \to \infty \end{cases}$ $\quad S_{11} = 0, \ S_{21} = \dfrac{ik-a}{ik+a}$

$S_{21} = 0, \ S_{2t} = S_{12}$

$S = \left(\dfrac{ik-a}{ik+a}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. For band stats, $k \to ik \ \rightarrow S = \left(\dfrac{-k-a}{-k+a}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$k = a, \quad E = -\dfrac{\hbar^2 k^2}{2m} = -\dfrac{\hbar^2 a^2}{2m}$ در توافق با ب